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Carathéodory Distances and Banach Algebras

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In recent years the Carathéodory pseudo-distance—together with its plurisubharmonicity properties—has found several applications to complex analysis in Banach spaces or, more in general, in locally convex vector spaces.

In the present paper the Carathéodory pseudo-distance and the corresponding infinitesimal pseudo-metric will be introduced in the framework of Banach algebras. In Section 2 a similar notion to that of spectral radius will be defined—in terms of Carathéodory's pseudo-distance, c_D, on a domain D of $\mathbb{C}$—for the spectra of elements of a complex Banach algebra $\mathscr{A}$, or, more in general, of a special kind of locally multiplicatively convex topological algebras. The new invariant, defined on the open set B consisting of those elements whose spectra belong to D, turns out to be logarithmically plurisubharmonic on B. This fact makes it possible to gather some information on the local behavior of the set-valued function mapping each element $x \in B$ into its spectrum, $\operatorname{Sp} x$. In this direction, some Liouville-type and Picard-type theorems are established. Furthermore, the relationship between the Carathéodory pseudo-distances on B and on D is investigated, in the case in which D is the unit disc of $\mathbb{C}$.

The final part of Section 2 is devoted to establishing a version, involving the Carathéodory pseudo-distance, of Wermer's classical subharmonicity theorem.

In Section 3 the Carathéodory pseudo-distance c_D will replace the euclidean distance in the definition of the nth diameter and of the transfinite diameter of compact subsets of D. The nth diameters ($n = 2, 3, \ldots, \infty$) of the spectrum $\operatorname{Sp} x$ of x will then be shown to be logarithmically plurisubharmonic functions of $x \in B$: a result which parallels recent theorems by Slodkowski and Aupetit. Here the proof appeals to the relationship between the spectral theory and Oka's theory of set-valued analytic functions, recently discovered by Slodkowski. If the Banach algebra is commutative, the proof becomes much easier and can be extended with no difficulty to joint spectra of finite sets of elements of B. The intrinsic character of the Carathéodory pseudo-distance yields the conclusion that the joint spectrum of elements of B, its polynomially convex hull and the Shilov boundary of the latter have the same n-th diameter for $n = 2, 3, \ldots, \infty$.

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Joint spectra are investigated also in Section 1, where a maximum principle for their polynomially convex hulls is established.

Some of the results of this work have been announced in [27].

1. A Maximum Theorem for Joint Spectra

1. The first applications of subharmonic functions to the theory of complex Banach algebras can be traced back in the following two statements.

Let $\mathscr{A}$ be a complex Banach algebra. For $x \in \mathscr{A}$, $\rho(x)$ will denote the spectral radius of x. Let U be a domain in $\mathbb{C}$, and let $f\colon U \to \mathscr{A}$ be a holomorphic map of U into $\mathscr{A}$.

THEOREM A. *The function* $\zeta \mapsto \log \rho(f(\zeta))$ *is subharmonic on* U [23, 24].

The monograph [2] by Aupetit gives, among other things, a detailed account of the applications of Theorem A, and of some of its consequences, to Banach algebras. More recent applications, due to Slodkowski [18, 19] and to Aupetit [3] will be referred to in subsections 7 – 9.

Let $\mathscr{B}$ be a uniform algebra on a compact Hausdorff space X, and let $M_{\mathscr{B}}$ be the maximal ideal space of $\mathscr{B}$. For $h \in \mathscr{B}$ and $\zeta \in \mathbb{C}$, let $h^{-1}(\zeta) = \{\chi \in M_{\mathscr{B}} : h(\chi) = \zeta\}$. Let U be a connected component of $\mathbb{C} \setminus f(X)$. Fix g in $\mathscr{B}$ and consider the set-valued function

$$\zeta \mapsto g(h^{-1}(\zeta)) \qquad (\zeta \in U).$$

For $\zeta \in U$, let

$$\mu_g(\zeta) = \max\{|\lambda| : \lambda \in g(h^{-1}(\zeta))\}.$$

THEOREM B. *The function* $\zeta \mapsto \log \mu_g(\zeta)$ *is subharmonic on* U [28].

This theorem has important applications, due to Aupetit and Wermer [4], to Bishop's classical theorem on the existence of analytic structures on $h^{-1}(U)$; cf. [2, 4, 29].

Theorem B has been extended by J. Wermer to maximum modulus algebras [30].

2. Assume $\mathscr{A}$ to have an identity. For $x \in \mathscr{A}$, let $\operatorname{Sp} x$ denote the spectrum of x. Examples given in [24] (cf. also [2]) show that the set-valued function

$$\zeta \mapsto \operatorname{Sp} f(\zeta) \qquad (\zeta \in U) \tag{2.1}$$

does not satisfy a maximum principle. In other words, the existence of $\zeta_0 \in U$ such that

$$\operatorname{Sp} f(\zeta) \subset \operatorname{Sp} f(\zeta_0) \qquad \text{for all } \zeta \in U,$$

does not always imply that the function (2.1) is constant.

Let $\widehat{\operatorname{Sp} x}$ be the complement of the unbounded component of $\mathbb{C} \setminus \operatorname{Sp} x$. It is easily seen (as a consequence of Lemma 3 of [24]; cf. Theorem 3, p. 12 of [2]) that the function

$$\zeta \mapsto \widehat{\operatorname{Sp} f(\zeta)} \qquad (\zeta \in U)$$

does satisfy a maximum principle.

The compact set $\widehat{\operatorname{Sp} f(\zeta)}$ is the polynomially convex hull of $\operatorname{Sp} f(\zeta)$.. Using this observation, the above conclusion wil now be extended to joint spectra of elements of a commutative complex Banach algebra. This result will be obtained without appealing to arguments involving Theorem A.

Let $\mathscr{A}$ be a commutative complex unital Banach algebra. For $x_1,\ldots, x_m$ in $\mathscr{A}$, $\sigma(x_1,\ldots, x_m) \subset \mathbb{C}^m$ will denote the joint spectrum of $x_1,\ldots, x_m$, and $\hat\sigma(x_1,\ldots, x_m)$ its polynomially convex hull; $\mathscr{P}(\hat\sigma(x_1,\ldots, x_m))$ will stand for the uniform closure, on the compact set $\hat\sigma(x_1,\ldots, x_m) \subset \mathbb{C}^m$, of the algebra of complex polynomials in m variables; $\Gamma(\hat\sigma(x_1,\ldots, x_m))$ will indicate the Shilov boundary of the algebra $\mathscr{P}(\hat\sigma(x_1,\ldots,x_m))$.

Let $f_j \colon U \to \mathscr{A}$ $(j = 1,\ldots, m)$ be holomorphic functions.

THEOREM I. *If*

$$\hat\sigma(f_1(\zeta),\ldots,f_m(\zeta)) \subset \hat\sigma(f_1(\zeta_0),\ldots,f_m(\zeta_0)) \tag{2.2}$$

for all $\zeta \in U$ and for some $\zeta_0 \in U$, then

$$\hat\sigma(f_1(\zeta),\ldots,f_m(\zeta)) = \hat\sigma(f_1(\zeta_0),\ldots,f_m(\zeta_0))$$

for all $\zeta \in U$.

The proof of the theorem will depend on the following two lemmas.

Let H be a compact set in $\mathbb{C}^m$ and let $\hat H$ be its polynomially convex hull.

LEMMA 2.1. *Every strong boundary point of $\mathscr{P}(\hat H)$ is a strong boundary point of $\mathscr{P}(H)$.*

Proof. Let z_0 be a strong boundary point of $\mathscr{P}(\hat H)$. Then $z_0 \in H$. If U is a neighborhood of z_0 in H there is a neighborhood V of z_0 in $\hat H$ such that

$V \cap H = U$. Since z_0 is a strong boundary point of $\mathscr{P}(\hat{H})$, there is $f \in \mathscr{P}(\hat{H})$ such that

$$\|f\|_V = f(z_0) = 1$$

and

$$|f(z)| < 1 \qquad \text{for all } z \in \hat{H} \setminus V.$$

Since $f_{|H} \in \mathscr{P}(H)$, and

$$1 = f_{|H}(z_0) \leqslant \|f_{|H}\|_U \leqslant \|f\|_V = 1,$$

then

$$\|f_{|H}\|_U = f_{|H}(z_0) = 1.$$

Being $H \setminus U \subset \hat{H} \setminus V$, then

$$|f_{|H}(z)| = |f(z)| < 1 \qquad \text{for all } z \in H \setminus U. \qquad \text{Q.E.D.}$$

Strong boundary points being dense in the Shilov boundary, Lemma 2.1 yields

$$\Gamma(\hat{H}) \subset \Gamma(H). \tag{2.3}$$

LEMMA 2.2. *If* (2.2) *holds, then*

$$\Gamma(\hat{\sigma}(f_1(\zeta_0),...,f_m(\zeta_0))) \subset \Gamma(\hat{\sigma}(f_1(\zeta),...,f_m(\zeta)))$$

for all $\zeta \in U$.

Proof. (a) It will be shown first that

$$\Gamma(\hat{\sigma}(f_1(\zeta_0),...,f_m(\zeta_0))) \subset \hat{\sigma}(f_1(\zeta),...,f_m(\zeta)) \qquad \text{for all } \zeta \in U.$$

Suppose there are $z_0 \in \mathbb{C}^m$ and $\zeta_1 \in U$ such that

$$z_0 \in \Gamma(\hat{\sigma}(f_1(\zeta_0),...,f_m(\zeta_0))), \qquad z_0 \notin \hat{\sigma}(f_1(\zeta_1),...,f_m(\zeta_1)).$$

Then there is an open neighborhood V of z_0 in $\mathbb{C}^m$, which is disjoint from $\hat{\sigma}(f_1(\zeta_1),...,f_m(\zeta_1))$. Since the Shilov boundary is the closure of the set of all strong boundary points, V contains a strong boundary point w_0 for $\mathscr{P}(\hat{\sigma}(f_1(\zeta_0),...,f_m(\zeta_0)))$. Let W be an open neighborhood of w_0 in V. There is some function $h \in \mathscr{P}(\hat{\sigma}(f_1(\zeta_0),...,f_m(\zeta_0)))$ such that

$$\|h\|_{\hat{\sigma}(f_1(\zeta_0),...,f_m(\zeta_0))} = h(w_0) = 1 \tag{2.4}$$

and

$$|h(z)| < 1 \qquad \text{for all } z \in \hat{\sigma}(f_1(\zeta_0),...,f_m(\zeta_0)) \setminus W. \tag{2.5}$$

Let $w_0^1,\dots,w_0^m$ be the coordinates of w_0 in $\mathbb{C}^m$, and let χ be a character of $\mathscr{A}$ such that $\chi(f_j(\zeta_0)) = w_0^j$ for $j = 1,\dots, m$. The function $g\colon U \to \mathbb{C}$ defined by

$$g(\zeta) = h(\chi(f_1(\zeta)),\dots, \chi(f_m(\zeta))) \qquad (\zeta \in U)$$

is holomorphic. Indeed, let $\{p_\nu\}$ be a sequence of polynomials converging uniformly to h on $\hat{\sigma}(f_1(\zeta_0),\dots,f_m(\zeta_0))$. If l is the boundary of a closed triangle contained in U then

$$\int_l g(\zeta)\, d\zeta = \lim_{\nu\to\infty} \int_l p_\nu(\chi(f_1(\zeta),\dots,f_m(\zeta)))\, d\zeta = 0,$$

and Morera's theorem yields the conclusion.

By (2.2) and (2.4),

$$|g(\zeta)| \leqslant 1 \qquad \text{for all } \zeta \in U,$$

and

$$g(\zeta_0) = 1.$$

Hence $g \equiv 1$. But this is a contradiction, for

$$\begin{aligned}(\chi(f_1(\zeta_1)),\dots, \chi(f_m(\zeta_1))) &\in \sigma(f_1(\zeta_1),\dots,f_m(\zeta_1)) \subset \hat{\sigma}(f_1(\zeta_1),\dots,f_m(\zeta_1))\\ &\subset \hat{\sigma}(f_1(\zeta_0),\dots,f_m(\zeta_0))\backslash W,\end{aligned}$$

and therefore, by (2.5),

$$|g(\zeta_1)| < 1.$$

(b) If z_0 is a strong boundary point for $\mathscr{P}(\hat{\sigma}(f_1(\zeta_0),\dots,f_m(\zeta_0)))$, then, by (a), $z_0 \in \hat{\sigma}(f_1(\zeta),\dots,f_m(\zeta))$ for any $\zeta \in U$. In view of (2.2), for any neighborhood L of z_0 in $\hat{\sigma}(f_1(\zeta),\dots,f_m(\zeta))$ there exists a neighborhood N of z_0 in $\hat{\sigma}(f_1(\zeta_0),\dots,f_m(\zeta_0))$ for which

$$L = N \cap \hat{\sigma}(f_1(\zeta),\dots,f_m(\zeta)).$$

There is a function $h_0 \in \mathscr{P}(\hat{\sigma}(f_1(\zeta_0),\dots,f_m(\zeta_0)))$ such that

$$\|h_0\|_{\hat{\sigma}(f_1(\zeta_0),\dots,f_m(\zeta_0))} = h_0(z_0) = 1$$

and

$$|h_0(z)| < 1$$

for all $z \in \hat{\sigma}(f_1(\zeta_0),\dots,f_m(\zeta_0))\backslash N$. By (2.2) the function $h = h_{0|\hat{\sigma}(f_1(\zeta),\dots,f_m(\zeta))}$ belongs to $\mathscr{P}(\hat{\sigma}(f_1(\zeta),\dots,f_m(\zeta))$, and

$$\|h\|_{\hat{\sigma}(f_1(\zeta),\dots,f_m(\zeta))} \leqslant \|h_0\|_{\hat{\sigma}(f_1(\zeta_0),\dots,f_m(\zeta_0))} = 1.$$

Since $h(z_0) = h_0(z_0) = 1$, then $\|h\|_{\hat{\sigma}(f_1(\zeta),\dots,f_m(\zeta))} = 1$.

If $z \in \hat{\sigma}(f_1(\zeta),\dots,f_m(\zeta))\backslash L$, then $z \in \hat{\sigma}(f_1(\zeta_0),\dots,f_m(\zeta_0))\backslash N$, and therefore $|h(z)| = |h_0(z)| < 1$.

In conclusion, if z_0 is a strong boundary point for $\mathscr{P}(\hat{\sigma}(f_1(\zeta_0),\dots,f_m(\zeta_0)))$, then z_0 is a strong boundary point for $\mathscr{P}(\hat{\sigma}(f_1(\zeta),\dots,f_m(\zeta)))$ for any $\zeta \in U$.

Since the strong boundary points are dense in the Shilov boundary, the proof of the lemma is complete. Q.E.D.

Proof of Theorem I. Let $z_0 \in \hat{\sigma}(f_1(\zeta_0),\dots,f_m(\zeta_0))$. Lemma 2.2 implies that for any $\zeta \in U$ and any complex polynomial p in m variables,

$$\begin{aligned}|p(z_0)| &\leqslant \|p\|_{\Gamma(\hat{\sigma}(f_1(\zeta_0),\dots,f_m(\zeta_0)))} \leqslant \|p\|_{\Gamma(\hat{\sigma}(f_1(\zeta),\dots,f_m(\zeta)))}\\ &\leqslant \|p\|_{\hat{\sigma}(f_1(\zeta),\dots,f_m(\zeta))}.\end{aligned}$$

Hence $z_0 \in \hat{\sigma}(f_1(\zeta),\dots,f_m(\zeta))$. Q.E.D.

COROLLARY 2.3. *If*

$$\sigma(f_1(\zeta),\dots,f_m(\zeta)) \subset \sigma(f_1(\zeta_0),\dots,f_m(\zeta_0)) \tag{2.6}$$

for all $\zeta \in U$ and for some $\zeta_0 \in U$, and if $\sigma(f_1(\zeta_0),\dots,f_m(\zeta_0))$ is polynomially convex, then

$$\hat{\sigma}(f_1(\zeta),\dots,f_m(\zeta)) = \sigma(f_1(\zeta_0),\dots,f_m(\zeta_0)) \quad \textit{for all } \zeta \in U.$$

In particular, if $\sigma(f_1(\zeta),\dots,f_m(\zeta))$ is polynomially convex for all $\zeta \in U$ and if (2.6) *holds, then*

$$\sigma(f_1(\zeta),\dots,f_m(\zeta)) = \sigma(f_1(\zeta_0),\dots,f_m(\zeta_0)) \qquad \textit{for all } \zeta \in U.$$

2. A CARATHÉODORY-TYPE SPECTRAL RADIUS

3. Let $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1.\}$ The Poincaré metric of Δ is the Riemannian metric expressed by

$$ds^2 = \frac{|d\zeta|^2}{(1-|\zeta|^2)^2}.$$

For $\tau \in \mathbb{C}$ and $\zeta \in \Delta$, let

$$\langle \tau \rangle_\zeta = \frac{|\tau|}{1-|\zeta|^2}$$

be the "length" of the vector τ at the point ζ. For ζ_1, ζ_2 on Δ, the Poincaré distance $\omega(\zeta_1, \zeta_2)$, i.e., the distance obtained by integration of the Poincaré metric, is given by

$$\omega(\zeta_1, \zeta_2) = \frac{1}{2} \log \frac{1+[\zeta_1, \zeta_2]}{1-[\zeta_1, \zeta_2]},$$

where

$$[\zeta_1, \zeta_2] = \left| \frac{\zeta_1 - \zeta_2}{1-\bar{\zeta}_1 \zeta_2} \right|. \tag{3.1}$$

The map $(\zeta_1, \zeta_2) \mapsto [\zeta_1, \zeta_2]$ is also a distance on Δ, whose infinitesimal form is the Poincaré metric, i.e.

$$\lim_{\tau \to 0} \frac{[\zeta, \zeta+\tau]}{|\tau|} = \langle 1 \rangle_\zeta \qquad \text{for all } \zeta \in \Delta.$$

Let D be a domain in a complex Banach space; c_D and k_D, γ_D and κ_D will denote respectively the Carathéodory and Kobayashi pseudo-distances, the Carathéodory and Kobayashi infinitesimal pseudo-metrics (for details, cf. [5]).

The following theorem has been proved in [26].

THEOREM C. *The function* $(x, y) \mapsto \log c_D(x, y)$ *is a plurisubharmonic function on* $D \times D$.

Remarks. 1. Since $c_\Delta = \omega$, then $(\zeta_1, \zeta_2) \mapsto \log \omega(\zeta_1, \zeta_2)$ is a plurisubharmonic function on $\Delta \times \Delta$. As a consequence, if f and g are holomorphic maps of Δ into Δ, then $\zeta \mapsto \log \omega(f(\zeta)\, g(\zeta))$ is a subharmonic function on Δ. Examples show that the function $\zeta \mapsto [f(\zeta), g(\zeta)]$ is not necessarily subharmonic [26].

2. As a consequence of theorem C, for any $x_0 \in D$, the function $x \mapsto \log c_D(x_0, x)$ is plurisubharmonic on D: a result previously established in [25].

The following proposition shows that $\omega(0, \cdot)$ is submultiplicative.

PROPOSITION 3.1. *For* ζ_1, ζ_2 *in* Δ

$$\omega(0, \zeta_1 \zeta_2) \leqslant \omega(0, \zeta_1)\, \omega(0, \zeta_2),$$

equality holding only when $\zeta_1 \zeta_2 = 0$, Since, by the Schwarz lemma,

$$\omega(0, \zeta_1 \zeta_2) \leqslant \min\{\omega(0, \zeta_1), \omega(0, \zeta_2)\} \qquad (\zeta_1, \zeta_2 \in \Delta),$$

the proposition is non-trivial only when $\omega(0, \zeta_1) < 1$ and $\omega(0, \zeta_2) < 1$.

Proof. For $0 < t_1 < 1$, let q be the C^∞ function on $[0, 1)$ defined by

$$q(t) = \omega(0, t_1)\, \omega(0, t) - \omega(0, t_1 t) \qquad (t \in [0, 1)).$$

Being $q(0) = 0$, the proof amounts to showing that $q(t) > 0$ whenever $0 < t < 1$. A direct computation yields

$$q'(t) = \frac{1}{(1 - t^2)(1 - t_1^2 t^2)} \times \left[t^2 (1 - t_1^2)\, t_1 + (1 - t_1^2 t^2) \sum_{\nu=1}^{+\infty} \frac{t_1^{2\nu+1}}{2\nu + 1} \right] > 0$$

for $0 \leqslant t < 1$. Thus q is strictly increasing on $[0,1)$. Q.E.D.

For any $\zeta \in \Delta$ and all positive integers ν_1 and ν_2

$$\omega(0, \zeta^{\nu_1 + \nu_2}) \leqslant \omega(0, \zeta^{\nu_1})\, \omega(0, \zeta^{\nu_2}),$$

and this implies [16] that the sequence $\{\omega(0, \zeta^\nu)^{1/\nu}\}$ $(\nu = 1, 2, \ldots)$ converges. The value of the limit is determined by the following

PROPOSITION 3.2. *The sequence* $\{\omega(0, \zeta^\nu)^{1/\nu}\}$ $(\nu = 1, 2, \ldots)$ *converges decreasingly to* $|\zeta|$.

Proof. For $0 < t < 1$ and $\nu \geqslant 1$

$$\omega(0, t^\nu) = \frac{1}{2} \log \frac{1 + t^\nu}{1 - t^\nu} = t^\nu (1 + s_\nu),$$

where

$$s_\nu = \sum_{n=1}^{+\infty} \frac{t^{2n\nu}}{2n + 1}.$$

Being $s_\nu > 0$, then $\omega(0, t^\nu) > t^\nu$, i.e.,

$$\omega(0, t^\nu)^{1/\nu} > t. \tag{3.2}$$

On the other hand

$$s_{\nu+1} < s_\nu. \tag{3.3}$$

Thus, for $\nu > 1$, $1 + s_\nu < 1 + s_1$, and therefore

$$(1 + s_\nu)^{1/\nu} < (1 + s_1)^{1/\nu}.$$

Since $\lim_{\nu\to\infty}(1 + s_1)^{1/\nu} = 1$, then

$$\lim_{\nu\to\infty} \omega(0, t^\nu)^{1/\nu} \leqslant t,$$

and (3.2) implies that equality holds. Since, by (3.3),

$$(1 + s_{\nu+1})^{1/(\nu+1)} < (1 + s_\nu)^{1/(\nu+1)} < (1 + s_\nu)^{1/\nu},$$

the convergence is decreasing. Q.E.D.

EXAMPLE. Let $\mathscr{A}$ be a complex unital Banach algebra. The domain

$$C = \{x \in \mathscr{A} : \operatorname{Sp} x \subset \Delta\}$$

is a balanced open neighborhood of 0 in $\mathscr{A}$. It is bounded if, and only if, there exists a constant $k > 0$ such that $k\rho(x) \geqslant \|x\|$ for all $x \in \mathscr{A}$. Hence [11] (or [2, p. 43]) if C is bounded, $\mathscr{A}$ is commutative. The converse is false as the example $\mathscr{A} = l^1(\mathbb{Z})$ shows. Up to re-normalization (by an equivalent norm) the unital Banach algebras for which C is bounded are all the function algebras.

It has been shown in [25] that, if $\mathscr{A}$ is any complex unital Banach algebra, then

$$c_C(0, x) \leqslant k_C(0, x) \leqslant \omega(0, \rho(x)) \qquad \text{for all } x \in C, \tag{3.4}$$

and that, if $\mathscr{A}$ is commutative, then both equalities hold.

A similar argument shows that

$$\gamma_C(0; u) \leqslant \kappa_C(0; u) \leqslant \rho(u) \qquad \text{for all } u \in \mathscr{A},$$

and that, if $\mathscr{A}$ is commutative, then both equalities hold.

Let $\mathscr{A}$ be commutative. By Proposition 3.1, for all x, y in C

$$\begin{aligned} c_C(0, xy) &= \omega(0, \rho(xy)) \leqslant \omega(0, \rho(x)\,\rho(y)) \leqslant \omega(0, \rho(x))\,\omega(0, \rho(y)) \\ &= c_C(0, x)\,c_C(0, y). \end{aligned}$$

By consequence, for all $x \in C$ and all positive integers ν_1 and ν_2,

$$c_C(0, x^{\nu_1+\nu_2}) \leqslant c_C(0, x^{\nu_1})\,c_C(0, x^{\nu_2}),$$

and therefore $\lim_{\nu\to\infty} c_C(0, x^\nu)^{1/\nu}$ exists. The limit can be computed by means of Proposition 3.2: since $\rho(x^\nu) = \rho(x)^\nu$, then

$$\lim_{\nu\to\infty} c_C(0, x^\nu)^{1/\nu} = \lim_{\nu\to\infty} \omega(0, \rho(x^\nu))^{1/\nu} = \lim_{\nu\to\infty} \omega(0, \rho(x)^\nu)^{1/\nu} = \rho(x),$$

i.e.,

$$\gamma_C(0, x) = \lim_{\nu\to\infty} c_C(0, x^\nu)^{1/\nu} \qquad \text{for all } x \in C.$$

Remark. If $\mathscr{A}$ is not commutative, then, by (3.4),

$$c_C(0, x^\nu)^{1/\nu} \leqslant k_C(0, x^\nu)^{1/\nu} \leqslant \omega(0, \rho(x^\nu))^{1/\nu} = \omega(0, \rho(x)^\nu)^{1/\nu}$$

for any $x \in C$ and any positive integer ν. Thus, by Proposition 3.2,

$$\limsup c_C(0, x^\nu)^{1/\nu} \leqslant \limsup k_C(0, x^\nu)^{1/\nu} \leqslant \rho(x).$$

4. Let $\mathscr{A}$ be a complex Banach algebra. For $x \in \mathscr{A}$, let E be a domain in $\mathbb{C}$ containing $\operatorname{Sp} x$. Given $\lambda_0 \in E$, let

$$\tau_E(\lambda_0, x) = \max\{c_E(\lambda_0, \lambda) : \lambda \in \operatorname{Sp} x\}.$$

If $E = \Delta$ and $\lambda_0 = 0$, $\tau_E(\lambda_0, x)$ is the hyperbolic spectral radius defined in [25]:

$$\tau_\Delta(0, x) = \omega(0, \rho(x)).$$

It is easily seen that

$$\tau_E(\lambda_0, x) = \sup\{\omega(0, \varphi(\lambda)) : \lambda \in \operatorname{Sp} x, \varphi \in \operatorname{Hol}(E, \Delta), \varphi(\lambda_0) = 0\} \tag{4.1}$$

(here $\operatorname{Hol}(E, \Delta)$ stands for the set of holomorphic maps of E into Δ). In fact, $\operatorname{Sp} x$ being compact and c_E being continuous on $E \times E$, there is some $\lambda_1 \in \operatorname{Sp} x$ such that $\tau_E(\lambda_0, x) = c_E(\lambda_0, \lambda_1)$. By Montel's theorem, there is some $\varphi_1 \in \operatorname{Hol}(E, \Delta)$ such that $\varphi_1(\lambda_0) = 0$ and $c_E(\lambda_0, \lambda_1) = \omega(0, \varphi_1(\lambda_1))$. For any $\varphi \in \operatorname{Hol}(E, \Delta)$, with $\varphi(\lambda_0) = 0$, and for any $\lambda \in \operatorname{Sp} x$,

$$\omega(0, \varphi(\lambda)) \leqslant c_E(\lambda_0, \lambda) \leqslant c_E(\lambda_0, \lambda_1) = \tau_E(\lambda_0, x),$$

and that yields (4.1).

Let $\varphi(x)$ be the element of $\mathscr{A}$ defined by φ and x (via Dunford's integral). By the spectral mapping theorem, (4.1) can be written

$$\tau_E(\lambda_0, x) = \sup\{\omega(0, \rho(\varphi(x))) : \varphi \in \operatorname{Hol}(E, \Delta), \varphi(\lambda_0) = 0\}. \tag{4.2}$$

Let $f: U \to \mathscr{A}$ be a holomorphic map of the domain $U \subset \mathbb{C}$ into $\mathscr{A}$, such that $\operatorname{Sp} f(\zeta) \subset E$ for all $\zeta \in U$. For any $\varphi \in \operatorname{Hol}(E, \Delta)$ with $\varphi(\lambda_0) = 0$, the function $\zeta \mapsto \varphi(f(\zeta))$ is a holomorphic map of U into $\mathscr{A}$ [5, pp. 30–32]. Thus, according to Proposition 5.4 of [25], the function

$$\zeta \mapsto \log \omega(0, \rho(\varphi(f(\zeta))))$$

is subharmonic on U. Since the function $\zeta \mapsto \tau_E(\lambda_0, f(\zeta))$ is upper semicontinuous, then (4.2) yields

THEOREM II. *The function $\zeta \mapsto \log \tau_E(\lambda_0, f(\zeta))$ is subharmonic on U.*

The above theorem can be extended to spectral sets as follows.

Let $\zeta_0 \in U$ and let $\operatorname{Sps}(f(\zeta_0))$ be a spectral set of $f(\zeta_0)$, i.e., an open and closed subset of $\operatorname{Sp} f(\zeta_0)$. If $\operatorname{Sps}(f(\zeta_0)) \neq \operatorname{Sp} f(\zeta_0)$, let E be an open connected neighborhood of $\operatorname{Sps}(f(\zeta_0))$ such that $\operatorname{Sp} f(\zeta_0) \cap E = \operatorname{Sps} f(\zeta_0))$. Let G be an open neighborhood of $\operatorname{Sp} f(\zeta_0) \setminus \operatorname{Sps}(f(\zeta_0))$ such that $E \cap G = \varnothing$. Let g be the holomorphic function defined on $E \cup G$ by $g(z) = z$ for all $z \in E$, $g(z) = a$ for all $z \in G$ and for some $a \in \operatorname{Sps}(f(\zeta_0))$, and let $g(f(\zeta_0))$ be the element of $\mathscr{A}$ defined by g and $f(\zeta_0)$. Then

$$\operatorname{Sps}(f(\zeta_0)) = \operatorname{Sp} g(f(\zeta_0)).$$

Thus, by the upper semicontinuity of the spectrum, there is a neighborhood V of ζ_0 in U such that the set $\operatorname{Sp} f(\zeta) \cap E$ is a non-empty compact subset of E, i.e., a non-empty spectral set of $f(\zeta)$, for all $\zeta \in V$.

Theorem II yields

PROPOSITION 4.1. *For any $\lambda_0 \in E$ the function*

$$\zeta \mapsto \log \max \{c_E(\lambda_0, \lambda): \lambda \in \operatorname{Sp} f(\zeta) \cap E\}$$

is subharmonic on V.

Going back to Theorem II, let $\mathscr{A}$ be unital and, for $R > 0$, let $U = \Delta_R = \{\zeta \in \mathbb{C}: |\zeta| < R\}$. Let $\lambda_0 = 0$ and $f(0) = 0$. For any $\varphi \in \operatorname{Hol}(E, \Delta)$ with $\varphi(0) = 0$, the holomorphic map $\zeta \mapsto \varphi(f(\zeta))$ is such that $\varphi(f(0)) = 0$ and $\rho(\varphi(f(\zeta))) < 1$. Hence [7, 25], $\rho(\varphi(f(\zeta))) \leqslant |\zeta|/R$ and therefore

$$\omega(0, \rho(\varphi(f(\zeta)))) \leqslant \omega\left(0, \frac{\zeta}{R}\right)$$

for all $\zeta \in \Delta_R$. Thus, by (4.2),

$$\tau_E(0, f(\zeta)) \leqslant \omega\left(0, \frac{\zeta}{R}\right) \qquad (\zeta \in \Delta_R).$$

If $f(0)=\lambda_0 1$ (where 1 denotes the identity of $\mathscr{A}$), the above argument can be applied to the holomorphic function $\zeta \mapsto f(\zeta)-\lambda_0 1$ and to the image of E by the translation $\lambda \mapsto \lambda-\lambda_0$, yielding

PROPOSITION 4.2. *If the holomorphic function $f: \Delta_R \to \mathscr{A}$ is such that $f(0)=\lambda_0 1$, then*

$$\tau_E(\lambda_0, f(\zeta)) \leqslant \omega\left(0, \frac{\zeta}{R}\right) \qquad \textit{for all } \zeta \in \Delta_R.$$

Letting $R \to \infty$, the above statement implies the following Carathéodory-type "Liouville theorem":

COROLLARY 4.3. *Let $f: \mathbb{C} \to \mathscr{A}$ be a holomorphic function such that $f(\zeta_0)$ is a scalar multiple of the identity for some $\zeta_0 \in \mathbb{C}$, and let E be a domain in $\mathbb{C}$ such that $\operatorname{Sp} f(\zeta) \subset E$ for all $\zeta \in \mathbb{C}$. If the Carathéodory pseudo-distance c_E is a distance, then $\operatorname{Sp} f(\zeta)$ consists of one point, which is independent of ζ.*

It is worth pointing out at this point how a "Picard-type" theorem can be established for a unital complex commutative Banach algebra $\mathscr{A}$.

Let D be a hyperbolic domain in $\mathbb{C}^m$ and let $f_j \in \operatorname{Hol}(\mathbb{C}, \mathscr{A})$ $(j=1,\ldots,m)$ be such that

$$\sigma(f_1(\zeta),\ldots,f_m(\zeta)) \subset D \qquad \text{for all } \zeta \in \mathbb{C}.$$

Since any holomorphic map $f: \mathbb{C} \to D$ is constant, for any character $\chi \in M_{\mathscr{A}}$ the point $(\chi \circ f_1(\zeta),\ldots,\chi \circ f_m(\zeta))$ is independent of ζ, i.e.,

$$\chi \circ (f_j(\zeta)-f_j(\zeta_0))=0 \qquad \text{for all } \zeta, \zeta_0 \in \mathbb{C},$$

and therefore

$$\sigma(f_1(\zeta)-f_1(\zeta_0),\ldots,f_m(\zeta)-f_m(\zeta_0))=\{0\} \qquad \text{for all } \zeta, \zeta_0 \in \mathbb{C}. \tag{4.3}$$

Since the complement of a pair of distinct points in $\mathbb{C}$ is a hyperbolic domain [8], (4.3) yields

LEMMA 4.4. *Let f be a holomorphic map of $\mathbb{C}$ into a complex unital abelian Banach algebra. If there are two distinct points a and b in $\mathbb{C}$, such that $\operatorname{Sp} f(\zeta) \cap \{a,b\}=\varnothing$ for all $\zeta \in \mathbb{C}$, then $f(\zeta)-f(\zeta_0)$ is topologically nilpotent for all $\zeta \in \mathbb{C}$ and for any choice of ζ_0. If moreover, for some $\zeta_0 \in \mathbb{C}$, $f(\zeta_0)$ is a scalar multiple of the identity then $\operatorname{Sp} f(\zeta)$ consists of one point which is independent of ζ.*

The extension of the above result to the non-commutative case remains an open question.

5. Turning the attention to another direction, it will be shown now how Theorem II can be extended to a large class of topological (not necessarily normed) algebras.

Let $\mathscr{A}$ be a complete, locally multiplicatively-convex topological algebra over $\mathbb{C}$; for definitions cf., e.g., [10]. By a result of Kaplansky, for every $x \in \mathscr{A}$, $\operatorname{Sp} x \neq \varnothing$. Let $\{p_i\}_{i\in I}$ be a family of submultiplicative pseudo-norms whose unit balls define a base for $\mathscr{A}$ [10, p. 9]. The null space $\mathscr{N}_i$ of p_i is a bilateral ideal of $\mathscr{A}$. Let $\mathscr{A}_i$ be the quotient algebra $\mathscr{A}_i = \mathscr{A}/\mathscr{N}_i$, and let $\Pi_i : \mathscr{A} \to \mathscr{A}_i$ be the natural homomorphism. Then $\mathscr{A}_i$ is a normed algebra for the norm $\bar{p}_i$ induced by p_i on $\mathscr{A}/\mathscr{N}_i$. Let $\overline{\mathscr{A}_i}$ be the completion of $\mathscr{A}_i$.

For every $x \in \mathscr{A}$, the spectrum $\operatorname{Sp}_{\mathscr{A}}$ is related to the spectrum $\operatorname{Sp}_{\overline{\mathscr{A}_i}} \Pi_i(x)$ of $\Pi_i(x)$ in $\overline{\mathscr{A}_i}$ by the formula

$$\operatorname{Sp}_{\mathscr{A}} x = \bigcup_i \operatorname{Sp}_{\overline{\mathscr{A}_i}} \Pi_i(x).$$

If $\rho_{\mathscr{A}}(x) = \sup\{|\lambda| : \lambda \in \operatorname{Sp}_{\mathscr{A}} x\}$, and $\rho_{\overline{\mathscr{A}_i}}(\Pi_i(x)) = \sup\{|\lambda| : \lambda \in \operatorname{Sp}_{\overline{\mathscr{A}_i}} \Pi_i(x)\}$ are the spectral radii of x and $\Pi_i(x)$, then [10, p. 19]

$$\rho_{\mathscr{A}}(x) = \sup_{i\in I} \{\rho_{\overline{\mathscr{A}_i}}(\Pi_i(x))\}.$$

Let f be a holomorphic map of U into $\mathscr{A}$ (i.e. a Gateaux analytic continuous map of U into $\mathscr{A}$). Then $\Pi_i \circ f \in \operatorname{Hol}(U, \overline{\mathscr{A}_i})$, and, by Theorem A, $\log \circ \rho_{\overline{\mathscr{A}_i}} \circ f$ is a subharmonic function on U. Since the upper envelope of a family of subharmonic functions is subharmonic if, and only if, it is upper semi-continuous, then the following lemma holds:

LEMMA 5.1. *Let $\mathscr{A}$ be such that the function $x \mapsto \operatorname{Sp} x$ is upper semi-continuous and that $\rho_{\mathscr{A}}(x) < \infty$ for every $x \in \mathscr{A}$. Let $f \in \operatorname{Hol}(U, \mathscr{A})$. The following conclusions hold*:

(a) *The function*

$$\log \circ \rho_{\mathscr{A}} \circ f \tag{5.1}$$

is subharmonic on U.

(b) *Let E be a domain in $\mathbb{C}$, and let f be such that $\operatorname{Sp} f(\zeta)$ is compact in E for all $\zeta \in U$. Then, for any $\lambda_0 \in E$, the function*

$$\zeta \mapsto \log \sup\{c_E(\lambda_0, \lambda) : \lambda \in \operatorname{Sp}_{\mathscr{A}} f(\zeta)\} \tag{5.2}$$

is subharmonic on U.

Part (a) follows from Theorem A, while part (b) can be established by a similar argument to that leading to Theorem II.

A class of topological algebras fulfilling the conditions in Lemma 5.1 consists of those complete, locally multiplicatively-convex algebras over $\mathbb{C}$ whose set of all quasi-regular elements is open. Let $\mathscr{A}$ be such an algebra. For every $x \in \mathscr{A}$, $\mathrm{Sp}_{\mathscr{A}} x$ is compact [10, p. 77]. Newburgh's proof of the upper semi-continuity of the spectrum in a Banach algebra can be easily adapted to establishing the following

LEMMA 5.2. *The map* $x \mapsto \mathrm{Sp}_{\mathscr{A}} x$ *is upper semi-continuous.*

Proof. If the conclusion is false, there exist: an element $x_0 \in \mathscr{A}$, a sequence $\{x_n\}$ converging to x_0, an open neighborhood V of $\mathrm{Sp}_{\mathscr{A}} x_0$ such that $\mathrm{Sp}_{\mathscr{A}} x_n \not\subset V$.

Let $\lambda_n \in \mathrm{Sp}\, x_n \backslash V$. If the sequence $\{\lambda_n\}$ is unbounded, there is a subsequence $\{\lambda_{n_j}\}$ diverging to ∞. Being

$$\lim_{n \to \infty} p_i(x_n - x_0) = 0 \qquad \text{for all indices } i,$$

then $\{p_i(x_n)\}$ is bounded for all i, and therefore

$$\lim_{j \to \infty} \frac{p_i(x_{n_j})}{\lambda_{n_j}} = 0 \qquad \text{for all } i,$$

i.e., the sequence $\{(1/\lambda_{n_j})\, x_{n_j}\}$ converges to 0. This is a contradiction, for 0 belongs to the open set of quasi-regular elements, whereas $(1/\lambda_{n_j})\, x_{n_j}$ is not quasi-regular. Hence the sequence $\{\lambda_n\}$ is bounded, and therefore contains a subsequence $\{\lambda_{n_j}\}$ converging to some $\lambda_0 \in \mathbb{C} \backslash V$.

From here the proof proceeds as in the case of Newburgh's theorem for Banach algebras (cf., e.g., [2, p. 7]). If $\lambda_0 = 0$, then $0 \notin \mathrm{Sp}_{\mathscr{A}} x_0$. Hence $\mathscr{A}$ is unital, and the sequence $\{\lambda_{n_j} 1 - x_{n_j}\}$ of non-invertible elements $\lambda_{n_j} 1 - x_{n_j}$ converges to the invertible element x_0. If $\lambda_0 \neq 0$, the quasi-regular element $(1/\lambda_0)\, x_0$ is the limit of the sequence $\{(1/\lambda_{n_j})\, x_{n_j}\}$ whose elements are not quasi-regular. This contradiction completes the proof of the lemma. Q.E.D.

In conclusion, the following proposition holds.

PROPOSITION 5.3. *Let* $\mathscr{A}$ *be a locally multiplicatively-convex algebra over* $\mathbb{C}$ *such that the set of all quasi-regular elements of* $\mathscr{A}$ *is open. For every* $f \in \mathrm{Hol}(U, \mathscr{A})$ *the functions* (5.1) *and* (5.2) *are subharmonic on* U.

6. This section is devoted to establishing an extension, involving the Carathéodory pseudo-distance, of Wermer's Theorem B. Let $\mathscr{A}$ be either a uniform algebra on a compact Hausdorff space or a maximum modulus algebra on a locally compact Hausdorff space. With the same notations as in Theorem B, the following lemma holds.

LEMMA 6.1. *If $g(h^{-1}(U)) \subset \Delta$, the function*

$$\zeta \mapsto \log \omega(0, \mu_g(\zeta))$$

is subharmonic on U.

Proof. The proof will consist in showing that, for every $a \in \mathbb{C}$, the function

$$\zeta \mapsto |e^{a\zeta}| \, \omega(0, \mu_g(\zeta)) \tag{6.1}$$

is subharmonic on U. This function has a power-series expansion converging at every point $\zeta \in U$

$$\begin{aligned} |e^{a\zeta}| \, \omega(0, \mu_g(\zeta)) &= |e^{a\zeta}| \sum_{\nu=0}^{+\infty} \frac{\mu_g(\zeta)^{2\nu+1}}{2\nu+1} \\ &= \sum_{\nu=0}^{\infty} \frac{1}{2\nu+1} (|e^{a\zeta/(2\nu+1)}| \, \mu_g(\zeta))^{2\nu+1}. \end{aligned}$$

By Theorem B,

$$\zeta \mapsto |e^{a\zeta/(2\nu+1)}| \, \mu_g(\zeta)$$

is subharmonic on U. Hence, also the function

$$\zeta \mapsto (|e^{a\zeta/(2\nu+1)}| \, \mu_g(\zeta))^{2\nu+1}$$

is subharmonic on U. Thus the function (6.1) is the pointwise limit of an increasing sequence of sub-harmonic functions. Since $\zeta \mapsto \mu_g(\zeta)$ is upper semi-continuous on U, the proof is complete. Q.E.D.

Now, let E be a domain in $\mathbb{C}$ such that $g(h^{-1}(U)) \subset E$. For $\lambda_0 \in E$, let

$$\sigma_E(\lambda_0, \zeta) = \max\{c_E(\lambda_0, \lambda) : \lambda \in g(h^{-1}(\zeta))\},$$

THEOREM III. *The function*

$$\zeta \mapsto \log \sigma_E(\lambda_0, \zeta)$$

is subharmonic on U.

Proof. Proceeding as in the proof of Theorem II, it can be shown that

$$\sigma_E(\lambda_0, \zeta) = \sup\{\omega(0, \mu_{\varphi \circ g}(\zeta)) : \varphi \in \mathrm{Hol}(E, \Delta), \varphi(\lambda_0) = 0\}.$$

Lemma 6.1 and the fact that $\zeta \mapsto \mu_g(\zeta)$ is upper semi-continuous yield the conclusion. Q.E.D.

3. Subharmonicity of Multidiameters

7. Let $\mathscr{A}$ be a unital complex Banach algebra. In the next three subsections the transfinite diameters, for the Carathéodory pseudo-distances, of the spectra of elements of $\mathscr{A}$ will be investigated.

A similar argument to Fekete's classical construction of the transfinite diameter of a compact set in the plane yields the following

Lemma 7.1. *Let X be a topological space and let δ be a continuous pseudo-distance on X. For any compact set $H \subset X$ and any integer $n \geqslant 2$, let*

$$\delta_n(H) = \max \left\{ \left(\prod_{i<j}^{1 \cdots n} \delta(x_i, x_j) \right)^{1/\binom{n}{2}} : x_i \in H \textit{ for } i = 1, \dots, n \right\}.$$

The sequence $\{\delta_n(H)\}$ is decreasing.

The limit

$$\delta_\infty(H) = \lim_{n \to \infty} \delta_n(H)$$

is, by definition, the *transfinite diameter* of H (with respect to δ), while $\delta_n(H)$ is called the n-th diameter of H (with respect to δ); $\delta_2(H)$ is the diameter of H.

In the case in which $X = \mathbb{C}$ and δ is the euclidean distance, $\delta_\infty(K)$ is Fekete's transfinite diameter, coinciding with the logarithmic capacity of H (cf., e.g., [22]).

Let $f\colon U \to \mathscr{A}$ be a holomorphic map of a domain $U \subset \mathbb{C}$ into $\mathscr{A}$. Let δ be the euclidean distance on $\mathbb{C}$. In [1] (cf. also [2]), Aupetit was able to prove, as a consequence of Theorem A, that the holomorphic function $\zeta \mapsto \log \delta_2(\operatorname{Sp} f(\zeta))$ is subharmonic on U. In [4] (cf. also [2]) Aupetit and Wermer proved, as a consequence of Theorem B, that $\zeta \mapsto \log \delta_2\, g(h^{-1}(\zeta))$ is a subharmonic function on U. Later on Slodkowski [19] extended those results proving, by an application of Theorems A and B, that the functions $\zeta \mapsto \log \delta_n(\operatorname{Sp} f(\zeta))$ and $\zeta \mapsto \log \delta_n(g(h^{-1}(\zeta))$ are subharmonic on U for $n = 2, 3, \dots, \infty$. In [18] Slodkowski established a connection between the set-valued functions $\zeta \mapsto \operatorname{Sp} f(\zeta)$ and $\zeta \mapsto g(h^{-1}(\zeta))$ and Oka's theory of pseudoconcave sets in $\mathbb{C}^2$ [18], whereby—as was noticed by Aupetit in [3]—the above statements concerning the nth diameters for the euclidean distance in $\mathbb{C}$ follow from a general theorem on subharmonicity of sections of pseudoconcave sets, due to Yamaguchi [31].

In the next section Yamaguchi's result will be extended to the case where nth diameters are computed in terms of the Carathéodory pseudo-distance in a domain. But, first, here is Oka's basic definition.

8. Let U be a domain in $\mathbb{C}$, and let $K: \zeta \mapsto K(\zeta)$ ($\zeta \in U$) be an upper semi-continuous set-valued function[1] such that $K(\zeta)$ is compact in $\mathbb{C}$ for every $\zeta \in U$. According to Oka [14], K is called an *analytic* multi-valued (or set-valued) function if the closed set $F = K(U) = \bigcup_{\zeta \in U} K(\zeta)$ is pseudo-concave, i.e., the open set

$$G = \{(\zeta, \lambda): \zeta \in U, \lambda \notin K(\zeta)\}$$

is a domain of holomorphy.

Let E be a domain in $\mathbb{C}$ containing $K(\zeta_0)$ for some $\zeta_0 \in U$. By the upper semi-continuity of K, there is a neighborhood of ζ_0 in U whose image by K is contained in E. Shrinking U if necessary, it will be assumed that $K(\zeta) \subset E$ for every $\zeta \in U$.

For $n = 2, 3, \ldots$ let $d_n(K(\zeta))$ be the nth diameter of $K(\zeta)$ with respect to the Carathéodory pseudo-distance c_E:

$$d_n(K(\zeta)) = \max \left\{ \left(\prod_{i<j}^{1 \cdots n} c_E(\lambda_i, \lambda_j) \right)^{1/\binom{n}{2}} : \lambda_i \in K(\zeta) \text{ for } i = 1, \ldots, n \right\}.$$

Recall that, if E is bounded, c_E is a distance defining the relative topology on E.

The function $\zeta \mapsto d_n(K(\zeta))$ is upper semi-continuous on U. Hence the transfinite diameter of $K(\zeta)$, $d_\infty(K(\zeta)) = \lim_{n \to \infty} d_n(K(\zeta))$, is an upper semi-continuous function on U.

In the case where $D = \mathbb{C}$ and the distance is the euclidean distance, Yamaguchi proved that the logarithm of the nth diameter of $K(\zeta)$ is a subharmonic function of $\zeta \in U$ for $n = 2, 3, \ldots, \infty$. This result will now be established for c_E.

THEOREM IV. *For* $n = 2, 3, \ldots, \infty$, *the function* $\zeta \mapsto \log d_n(K(\zeta))$ *is subharmonic on* U.

Proof. (a) It has been shown in [31, pp. 420–421] that there is no restriction in assuming that F satisfies the following condition:

(L) For any $\zeta_0 \in U$ and any $\lambda_0 \in K(\zeta_0)$ there is some $r > 0$ and a scalar-valued holomorphic function ψ on the disc $B = \{\zeta \in \mathbb{C}: |\zeta - \zeta_0| < r\}$ such that $B \subset U$, $\psi(\zeta_0) = \lambda_0$ and $\psi(\zeta) \in K(\zeta)$ for all $\zeta \in B$.

In fact, as a consequence of the solution of the "boundary problem" for plurisubharmonic functions (cf. [15]), for any domain $U' \Subset U$ there is a sequence $\{F_\nu\}$ of pseudo-concave sets F_ν in $\mathbb{C}^2$ such that:

for any ν and for any $\zeta \in U'$ the set $K_\nu(\zeta) = \{(\zeta, \lambda) \in \mathbb{C}^2: \lambda \in F_\nu\}$ is compact and upper semi-continuous with respect to ζ;

[1] For every neighborhood W of $K(\zeta_0)$ ($\zeta_0 \in U$) there is a neighborhood V of ζ_0 in U such that $K(\zeta) \subset W$ for all $\zeta \in V$.

$$K_\nu(\zeta) \subset E \qquad \text{for all } \zeta \in U';$$

$$F_\nu \supset F_{\nu+1};$$

$$\cap F_\nu = F; \qquad \text{for any } \nu,\ F_\nu \text{ satisfies condition } L).$$

Once Theorem IV is proved for the functions $\zeta \mapsto \log d_n(K_\nu(\zeta))$ $(n = 2, 3, \ldots)$, then it follows for $K(\zeta)$, since the sequence $\{d_n(K_1(\zeta)), d_n(K_2(\zeta)), \ldots\}$ converges decreasingly to $d_n(K(\zeta))$ for $n = 2, 3, \ldots$.
It will be assumed henceforth that condition (L) is fulfilled.

(b) Let $n \geqslant 2$ be finite. Since $K(\zeta_0)$ is compact and c_E is continuous there are points $\lambda_1^0, \ldots, \lambda_n^0$ in $K(\zeta_0)$ such that

$$d_n(K(\zeta_0))^{\binom{n}{2}} = \prod_{i<j}^{1\cdots n} c_E(\lambda_i^0, \lambda_j^0).$$

Taking into account condition (L), let $\psi_1, \ldots, \psi_n$ be scalar valued holomorphic functions on $B = \{\zeta \in \mathbb{C} : |\zeta - \zeta_0| < r\}$ such that $B \subset U$, $\psi_j(\zeta_0) = \lambda_j^0$, $\psi_j(\zeta) \in K(\zeta)$ for any $\zeta \in B$ and $j = 1, \ldots, n$. The function

$$\zeta \mapsto \log \left(\prod_{i<j}^{1\cdots n} c_E(\psi_i(\zeta), \psi_j(\zeta)) \right)^{1/\binom{n}{2}}$$

is subharmonic on B, and its value at ζ_0 is $\log d_n(K(\zeta_0))$. Thus, for $0 < s < r$

$$\begin{aligned} \log d_n(K(\zeta_0)) \leqslant \frac{1}{2\pi} \int_0^{2\pi} \log \Bigg(\prod_{i<j}^{1\cdots n} c_E(\psi_i(\zeta_0 + s e^{\sqrt{-1}\theta}),& \\ \psi_j(\zeta_0 + s e^{\sqrt{-1}\theta})) \Bigg)^{1/\binom{n}{2}} d\theta.& \end{aligned} \tag{8.1}$$

Since $\zeta \mapsto d_n(K(\zeta))$ is upper semi-continuous, and

$$\left(\prod_{i<j}^{1\cdots n} c_E(\psi_i(\zeta), \psi_j(\zeta)) \right)^{1/\binom{n}{2}} \leqslant d_n(K(\zeta))$$

for all $\zeta \in B$, (8.1) implies that

$$\log d_n(K(\zeta_0)) \leqslant \frac{1}{2\pi} \int_0^{2\pi} \log d_n(K(\zeta_0 + s e^{\sqrt{-1}\theta}))\, d\theta$$

for $0 < s < r$, proving thereby that the function $\zeta \mapsto \log d_n(K(\zeta))$ is subharmonic on U for $n = 2, 3, \ldots$.

(c) Since $d_n(K(\zeta)) \searrow d_\infty(K(\zeta))$ as $n \to \infty$ for all $\zeta \in U$, also the function $\zeta \mapsto \log d_\infty(K(\zeta))$ is subharmonic on U. Q.E.D.

Using condition (L) the following proposition can be proved following an argument similar to the one developed in (b):

PROPOSITION 8.1. *For any point $\lambda_0 \in E$ the function*

$$\zeta \mapsto \log \max\{c_E(\lambda_0, \lambda) : \lambda \in K(\zeta)\}$$

is subharmonic on U.

In the case where $E = \mathbb{C}$ and the distance is the euclidean distance, Proposition 8.1 was proved by Nishino in [13, p. 230].

Remark. In [21] Tsuji introduced the transfinite diameter of a compact subset of Δ for the distance defined by (3.1). Due to Remark 1 following Theorem C, Theorem IV is false for the transfinite diameter defined by Tsuji.

9. The following theorem has been proved by Z. Slodkowski in [18].

THEOREM D. (a) *If $\mathscr{A}$ is a complex unital Banach algebra and $f: U \to \mathscr{A}$ is a holomorphic map of a domain $U \subset \mathbb{C}$ into $\mathscr{A}$, the set-valued function $\zeta \mapsto \operatorname{Sp} f(\zeta)$ is analytic.*

(b) *If $\mathscr{A}$ is a uniform algebra, the set-valued function $\zeta \mapsto g(h^{-1}(\zeta))$ is analytic on U.*

As a consequence of Theorem IV and of Theorem D, the following statement holds, where E is a domain in $\mathbb{C}$ such that $\operatorname{Sp} f(\zeta) \subset E$ and $g(h^{-1}(\zeta)) \subset E$ for all $\zeta \in U$.

THEOREM V. *The functions $\zeta \mapsto \log d_n(\operatorname{Sp} f(\zeta))$ and $\zeta \mapsto \log d_n(g(h^{-1}(\zeta)))$ are subharmonic on U for $n = 2, 3, ..., \infty$.*

Theorem D and Proposition 8.1 provide also new proofs of Theorems II and III. However, these latter proofs involve condition (L), which in turn depends on deep results of Oka [15], while the arguments given in subsections 4, 6 are more elementary. It would be interesting to give a self-contained direct proof (avoiding condition (L)) also for Theorem V, as it was done by Slodkowski in [19] in the case of the euclidean distance on $E = \mathbb{C}$.

A proof of this type is easy for commutative algebras. In fact more can be said in this case.

Going back to subsection 1, let $\mathscr{A}$ be a commutative complex unital Banach algebra and let $M_{\mathscr{A}}$ be the maximal ideal space of $\mathscr{A}$ endowed with the Gel'fand topology. As in subsection 1, let $f_1, ..., f_m$ be holomorphic maps of a domain U of $\mathbb{C}$ into $\mathscr{A}$, and for $\zeta \in U$ let $\sigma(f_1(\zeta), ..., f_m(\zeta)) \subset \mathbb{C}^m$ be the joint spectrum of $f_1(\zeta),\ldots,f_m(\zeta)$. Let D be a domain in $\mathbb{C}^m$ for which $\sigma(f_1(\zeta),\ldots,f_m(\zeta)) \subset D$ for all $\zeta \in U$, and let c_D be the Carathéodory pseudo-distance in D. Denoting by $d_n(\sigma(f_1(\zeta),\ldots,f_m(\zeta)))$ the nth diameter of $\sigma(f_1(\zeta),\ldots,f_m(\zeta))$ $(n = 2, 3,\ldots)$ for c_D, then

$$d_n(\sigma(f_1(\zeta),\ldots,f_m(\zeta))) = \max \left\{ \left(\prod_{i<j}^{1\cdots n} c_D((\chi_i(f_1(\zeta)),\ldots,\chi_i(f_m(\zeta))), (\chi_j(f_1(\zeta)),\ldots,\chi_j(f_m(\zeta)))) \right)^{1/\binom{n}{2}} : \chi_j \in M_{\mathscr{A}} \text{ for } j = 1,\ldots,n \right\}.$$

Since the function

$$\zeta \mapsto \log c_D((\chi_i(f_1(\zeta)),\ldots,\chi_i(f_m(\zeta))), (\chi_j(f_1(\zeta)),\ldots,\chi_j(f_m(\zeta))))$$

is subharmonic on U by Theorem C, and since $\zeta \mapsto d_n(\sigma(f_1(\zeta),\ldots,f_m(\zeta)))$ is upper semi-continuous on U, the following theorem holds.

THEOREM VI. *The function* $\zeta \mapsto \log d_n(\sigma(f_1(\zeta),\ldots,f_m(\zeta)))$ *is subharmonic on* U *for* $n = 2, 3,\ldots, \infty$.

The conclusion concerning $n = \infty$ follows, as before, from the fact that

$$d_n(\sigma(f_1(\zeta),\ldots,f_m(\zeta))) \searrow d_\infty(\sigma(f_1(\zeta),\ldots,f_m(\zeta))) \quad \text{as} \quad n \to \infty.$$

It will be shown in the next subsection that, if D is a domain of holomorphy in $\mathbb{C}^m$, then

$$\begin{aligned} d_n(\sigma(f_1(\zeta),\ldots,f_m(\zeta))) &= d_n(\hat{\sigma}(f_1(\zeta),\ldots,f_m(\zeta))) \\ &= d_n\Gamma(\sigma(f_1(\zeta),\ldots,f_m(\zeta))) = d_n(\Gamma(\hat{\sigma}(f_1(\zeta),\ldots,f_m(\zeta))) \end{aligned} \tag{9.1}$$

for $n = 2,\ldots, \infty$.

10. Let K be a compact polynomially convex set in $\mathbb{C}^m$. As in subsection 2 $\mathscr{P}$, $\mathscr{P}(K)$ will denote the uniform closure on K of the algebra of polynomials in m variables. The compact set K being polynomially convex, the maximal ideal space of $\mathscr{P}(K)$ is K itself. Let Γ be the Shilov boundary of $\mathscr{P}(K)$.

Let K_0 be a connected component of K. Then K_0 is open and closed for the relative topology in K. As a consequence of H. Rossi's local maximum modulus principle, [9, p. 194],

$$K_0 \cap \Gamma \neq \varnothing. \tag{10.1}$$

Let D be a connected open neighborhood of K in $\mathbb{C}^m$, and assume D to be a domain of holomorphy in $\mathbb{C}^m$.

PROPOSITION 10.1. *Let ϕ be a continous plurisubharmonic function on D. Then $\phi_{|K}$ reaches its maximum on Γ.*

Proof. By the Oka–Weil theorem [6, p. 84], the uniform closure on K of the algebra $\mathscr{H}(D)$ of all holomorphic functions on D is $\mathscr{P}(K)$, and

$$K=\hat{K}_{\mathscr{P}(K)}=\hat{\Gamma}_{\mathscr{P}(K)}=\hat{\Gamma}_{\mathscr{H}(D)}=\{x\in D\colon |f(x)|\leqslant \|f\|_\Gamma\ \forall f\in \mathscr{H}(D)\}.$$

Let $a>\|\phi\|_\Gamma$. The set

$$D_a=\{x\in D\colon \phi(x)<a\}$$

is a domain of holomorphy, which is Runge in D [12]. This latter fact implies that the uniform closure of $\mathscr{H}(D_a)_{|\Gamma}$ is the same as the uniform closure of $\mathscr{H}(D)_{|\Gamma}$. Moreover

$$\begin{aligned}\hat{\Gamma}_{\mathscr{H}(D_a)}&=\hat{\Gamma}_{\mathscr{H}(D)_{|D_a}}=\{x\in D_a\colon |f(x)|\leqslant \|f\|_\Gamma\ \forall f\in\mathscr{H}(D)\}\\&=\hat{\Gamma}_{\mathscr{H}(D)}\cap D_a=K\cap D_a.\end{aligned}$$

The domain D_a being a domain of holomorphy, $\hat{\Gamma}_{\mathscr{H}(D_a)}$ is compact in D_a. Therefore

$$K\cap D_a \text{ is compact in } D_a \text{ for all } a>\|\phi\|_\Gamma. \tag{10.2}$$

For $x_0\in K$, let K_0 be the connected component of K containing x_0.

Conditions (10.1) and (10.2), together with the continuity of ϕ imply that $K_0\subset D_a$, i.e.,

$$\phi(x_0)<a \text{ for every } a>\|\phi\|_\Gamma. \qquad \text{Q.E.D.}$$

PROPOSITION 10.2. *For any $n=2,3,\ldots,\infty$*

$$d_n(K)=d_n(\Gamma).$$

Proof. For $n=2,3,\ldots$, let $\lambda_1^0,\ldots,\lambda_n^0$ be points of K such that

$$d_n(K)^{\binom{n}{2}}=\prod_{i<j}^{1\cdots n} c_D(\lambda_i^0,\lambda_j^0).$$

The proposition will be proved by showing that, for any $i_0=1,\ldots,n$, some point $\lambda_{i_0}^1\in\Gamma$ exists such that

$$d_n(K)^{\binom{n}{2}}=\prod_{i<j}^{1\cdots n} c_D(\mu_i^0,\mu_j^0)$$

with $\mu_j^0=\lambda_j^0$ for $i=1,\ldots,\hat{i}_0,\ldots,n$, $\mu_{i_0}^0=\lambda_{i_0}^1$.

There is no restriction in assuming $i_0 = 1$. Consider the function

$$\phi: \lambda \mapsto c_D(\lambda, \lambda_2^0)\, c_D(\lambda, \lambda_3^0) \cdots c_D(\lambda, \lambda_n^0).$$

By Theorem C, the function ϕ is continuous and plurisubharmonic on D. Proposition 10.1 yields the conclusion for the nth diameters, and then, letting $n \to \infty$, for the transfinite diameters. Q.E.D.

Remarks. 1. For $m = 1$, the above proposition follows directly from the maximum principle.

2. For $n = 2$, i.e., for the diameter of K, the condition that D be a domain of holomorphy is not required.

In fact, let D be any domain in $\mathbb{C}^m$ containing K. Then there exist two points λ_1^0, λ_2^0 in K, and a function $g \in \mathrm{Hol}(D, \Delta)$ such that $g(\lambda_1^0) = 0$,

$$\begin{aligned} d_2(K) = c_D(\lambda_1^0, \lambda_2^0) = \omega(0, g(\lambda_2^0)) &= \frac{1}{2} \log \frac{1 + |g(\lambda_2^0)|}{1 - |g(\lambda_2^0)|} \\ &= \max \left\{ \frac{1}{2} \log \frac{1 + |g(\lambda)|}{1 - |g(\lambda)|} : \lambda \in K \right\}. \end{aligned}$$

By the Oka–Weil theorem, $g_{|K} \in \mathscr{P}(K)$ [6, p. 84]. Thus $g_{|K}$ reaches its maximum on Γ.

Now, let K be a compact set in $\mathbb{C}^m$. Let $\hat{K}$ be its polynomially convex hull in $\mathbb{C}^m$, and let $\Gamma(K)$, $\Gamma(\hat{K})$ be the Shilov boundaries of $\mathscr{P}(K)$ and $\mathscr{P}(\hat{K})$. Then

$$\Gamma(\hat{K}) \subset \Gamma(K).$$

Let D be a domain of holomorphy in $\mathbb{C}^m$ such that $\hat{K} \subset D$.
Since $K \subset \hat{K}$, then for any $n = 2, 3, \ldots, \infty$

$$d_n(\Gamma(\hat{K})) \leqslant d_n(\Gamma(K)) \leqslant d_n(K) \leqslant d_n(\hat{K}).$$

Proposition 10.2 implies therefore that

$$d_n(\Gamma(\hat{K})) = d_n(\Gamma(K)) = d_n(K) = d_n(\hat{K}) \qquad \text{for any } n = 2, 3, \ldots, \infty.$$

That proves (9.1).

Note added in proof. In a forthcoming article Proposition 8.1 has been extended in the following form. Let E be a domain in $\mathbb{C}$ and let h_E be the Hausdorff pseudo-distance computed in E in terms of the Carathéodory pseudo-distance c_E. Let K_1 and K_2 be two set-valued analytic functions defined on a domain $U \subset \mathbb{C}$, such that $K_j(\zeta) \subset E$ for all $\zeta \in U$ and $j = 1, 2$. Then the function

$$\zeta \mapsto \log h_E(K_1(\zeta), K_2(\zeta))$$

is subharmonic on U.

In the same article, Theorem I has been extended to joint spectra of any (not necessarily finite) family of elements in a complex unital abelian Banach algebra.

REFERENCES

1. B. AUPETIT, Caractérisation spectrale des algèbres de Banach commutatives, *Pacific J. Math.* **63** (1976), 23–35.
2. B. AUPETIT, "Propriétés spectrales des algèbres de Banach," Lecture Notes in Mathematics No. 735, Springer-Verlag, Berlin/Heidelberg/New York, 1979.
3. B. AUPETIT, Analytic multivalued functions in Banach algebras and uniform algebras, *Advances in Math.* **44** (1982), 18–60.
4. B. AUPETIT AND J. WERMER, Capacity and uniform algebras, *J. Functional Anal.* **28** (1978), 386–400.
5. T. FRANZONI AND E. VESENTINI, "Holomorphic Maps and Invariant Distances," North-Holland Mathematics Studies No. 40, North-Holland, Amsterdam/New York/Oxford, 1980.
6. T. W. GAMELIN, "Uniform Algebras," Prentice–Hall, Englewood Cliffs, N.J., 1969.
7. J. GLOBEVNIK, Schwarz's lemma for the spectral radius, *Rev. Roumaine Math. Pures Appl.* **19** (1974), 1009–1012.
8. S. KOBAYASHI, "Hyperbolic Manifolds and Holomorphic Maps," Marcel Dekker, New York, 1970.
9. G. M. LEIBOWITZ, "Lectures on Complex Function Algebras," Scott–Foresman, Glenview, Ill., 1970.
10. E. MICHAEL, Locally multiplicatively convex topological algebras, *Mem. Amer. Math. Soc.* **11** (1952).
11. C. LE PAGE, Sur quelques conditions entraînant la commutativité dans les algèbres de Banach, *C.R. Acad. Sci. Paris, Sér. A–B* **265** (1967), 235–237.
12. R. NARASIMHAN, The Levi problem for complex spaces, II, *Math. Ann.* **146** (1962), 195–216.
13. T. NISHINO, Sur les ensembles pseudoconcaves, *J. Math. Kyoto Univ.* **1–2** (1962), 225–245.
14. K. OKA, Note sur les familles des fonctions analytiques multiformes etc., *J. Sci. Hiroshima Univ.* **4** (1934), 93–98.
15. K. OKA, Domaines finis sans point critique intérieur, *Japan. J. Math.* **27** (1953), 97–155; reprinted in K. Oka, "Sur les fonctions analytiques de plusieurs variables," Iwanami Shoten, Tokyo, 1961, pp. 158–234.
16. G. POLYA AND G. SZEGÖ, "Aufgaben und Lehrsätze aus der Analysis, Vol. I," Springer, Berlin, 1925.
17. V. N. SENICKIN, Subharmonic functions and analytic structure in the maximal ideal space of a uniform algebra, *Math. USSR Sbornik* **36** (1980), 111–126.
18. Z. SLODKOWSKI, Analytic set-valued functions and spectra, *Math. Ann.* **256** (1981), 363–386.
19. Z. SLODKOWSKI, On subharmonicity of the capacity of the spectrum, *Proc. Amer. Math. Soc.* **81** (1981), 243–249.

20. E. L. STOUT, "The Theory of Uniform Algebras," Bodgen and Quigley, New York, 1971.
21. M. TSUJI, Some metrical theorems on Fuchsian groups, *Japan. J. Math.* **19** (1947), 483–516.
22. M. TSUJI, "Potential Theory in Modern Function Theory," 2nd ed., Chelsea, New York, 1975.
23. E. VESENTINI, On the subharmonicity of the spectral radius, *Boll. Un. Mat. Ital.* (4) **1** (1968), 427–429.
24. E. VESENTINI, Maximum theorems for spectra, *in* "Essays on topology and related topics," Mémoires dédiés à Georges de Rham, pp. 111–117; Springer-Verlag, Berlin/Heidelberg/New York, 1970.
25. E. VESENTINI, Variations on a theme of Carathéodory, *Ann. Scuola Norm. Sup. Pisa*, (4) **6** (1979), 39–68.
26. E. VESENTINI, "Complex geodesics and holomorphic maps," Symposia Math., XXVI (1982), 211–230.
27. E. VESENTINI, Subharmonicity in Banach algebras, *Proc. Steklov Inst. Math.*, to appear.
28. J. WERMER, Subharmonicity and hulls, *Pacific J. Math.* **58** (1975), 283–290.
29. J. WERMER, Capacity and uniform algebras, *in* "Proceedings of Symposia in Pure Mathematics," Vol. 35, Part. I, pp. 445–449, Amer. Math. Soc., Providence, Rhode Island, 1979.
30. J. WERMER, Maximum modulus algebras and singularity sets, *Proc. Roy. Soc. Edinburgh Sect. A*, **86** (1980), 327–331.
31. H. YAMAGUCHI, Sur une uniformité des surfaces constantes d'une fonction entière de deux variables complexes, *J. Math. Kyoto Univ.* **13–3** (1973), 417–433.